

EXTREMAL CONTRACTIONS OF 2-FANO FOURFOLDS

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ABSTRACT. We consider extremal contractions on smooth Fano fourfolds whose second Chern character is positive. We show that such contractions can neither be of fiber type nor contract a divisor to a point.

1. INTRODUCTION

A Fano manifold X is called *2-Fano*, if its second Chern character $\mathrm{ch}_2(X) := \mathrm{ch}_2(T_X)$ is positive, i.e., $\mathrm{ch}_2(X) \cdot S > 0$ for all $S \in \overline{NE}_2(X) \setminus \{0\}$. We recall that for a vector bundle E , its second Chern character $\mathrm{ch}_2(E)$ is given by

$$\mathrm{ch}_2(E) = \frac{1}{2}(c_1^2(E) - 2c_2(E)).$$

2-Fano manifolds have first been introduced by de Jong and Starr in [dJS06] and [dJS07] in order to obtain a natural sufficient condition for the technical notion of *rational simple connectedness*. The property of rational simple connectedness is used in [dJHS11] to prove a generalization to the surface case of the Graber–Harris–Starr theorem on the existence of sections in families of rationally connected varieties over curves [GHS03].

The classification of 2-Fano manifolds is still very much an open problem. In dimension 3, as shown by Araujo–Castravet in [AC12], the only 2-Fano manifolds are \mathbb{P}_3 and Q_3 . For $n := \dim X \geq 4$, [AC12] gives a classification of 2-Fano manifolds with index $\geq n - 2$, but for arbitrary index, the overall picture remains unclear.

All known examples of 2-Fanos seem to have second Betti number one, so a natural question to ask is whether it is possible for a 2-Fano manifold X to have $b_2(X) \geq 2$.

The present paper studies this question in dimension 4, i.e., we let X be a 2-Fano fourfold and assume that $b_2(X) \geq 2$. Then, by Mori theory, there exists an extremal contraction

$$f: X \rightarrow Y,$$

where Y is a normal projective variety with $1 \leq \dim Y \leq 4$.

We first study the cases where $\dim Y < 4$. In section 2 we exclude the case $\dim Y = 3$ by showing that a 2-Fano manifold X of arbitrary dimension cannot have an extremal contraction to a variety of dimension $\dim X - 1$ (cf. Proposition 2.2). Sections 3 and 4 deal with the cases $\dim Y = 2$ and $\dim Y = 1$, respectively. We thus arrive at the following theorem:

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Theorem 1. *Let X be a smooth 2-Fano fourfold and $f: X \rightarrow Y$ the contraction of an extremal ray in $\overline{NE}(X)$. Then f is birational.*

Now in the birational case, the situation is more complicated, so we cannot give a complete result. Nevertheless, we do some calculations for blow-ups in section 5 and can thus rule out one important case of birational contractions:

Theorem 2. *Let X be a smooth 2-Fano fourfold and $f: X \rightarrow Y$ a birational contraction of an extremal ray in $\overline{NE}(X)$. Then f cannot contract a divisor to a point.*

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2. CONIC BUNDLES

Let Z, U be smooth varieties and $g: Z \rightarrow U$ a conic bundle (i.e., a proper morphism such that each fiber of g is isomorphic to a conic in \mathbb{P}_2). Then there exists an effective divisor $\Delta_g \subset U$ such that $Z_u := g^{-1}(u)$ is smooth if and only if $u \notin \Delta_g$.

Lemma 2.1. *In the situation given above, for any smooth proper curve $C \subset U$ such that $S := g^{-1}(C)$ is smooth, we have*

$$\mathrm{ch}_2(T_Z|_S) = -\frac{3}{2} \deg \Delta_g|_C \leq 0.$$

Proof. Since g is flat, we have $N_{S|Z}^* = (g|_S)^* N_{C|U}^*$ and thus $\mathrm{ch}_2(N_{S|Z}^*) = (g|_S)^* \mathrm{ch}_2(N_{C|U}^*)$, which is zero because $\dim C = 1$. So from the exact sequence

$$0 \longrightarrow T_S \longrightarrow T_Z|_S \longrightarrow N_{S|Z} \longrightarrow 0,$$

we obtain

$$\mathrm{ch}_2(T_Z|_S) = \mathrm{ch}_2(S).$$

This shows that we can assume from now on that $U = C$ and $Z = S$.

We now consider the rank-3 vector bundle $E := g_* \mathcal{O}_Z(-K_Z)$ and the projective bundle

$$\pi: \mathbb{P}(E) \rightarrow U.$$

An easy calculation shows that Z can be embedded into $\mathbb{P}(E)$ as a divisor

$$Z \in |\mathcal{O}_{\mathbb{P}(E)}(2) + \pi^*(\det E^* - K_U)|.$$

Equivalently, Z is given by a section

$$s \in H^0(S^2 E \otimes \det E^* \otimes \mathcal{O}_U(-K_U)) \subset \mathrm{Hom}(E^*, E \otimes \det E^* \otimes \mathcal{O}_U(-K_U)).$$

From this, it follows that

$$\Delta_g \in |\det E^* - 3K_U|.$$

On the other hand, by [dJS06, Lem. 4.1], we have

$$\mathrm{ch}_2(\mathbb{P}(E)) = \frac{3}{2} \zeta^2 + \pi^* c_1(E^*) \cdot \zeta,$$

where $\xi := c_1(\mathcal{O}_{\mathbb{P}(E)}(1))$. It follows that

$$\begin{aligned} \text{ch}_2(Z) &= (\text{ch}_2(\mathbb{P}(E)) - \frac{1}{2}(2\xi + \pi^*c_1(E^*) + \pi^*c_1(U))^2)|_Z \\ &= (-\frac{1}{2}\xi^2 - \pi^*c_1(E^*)\cdot\xi - 2\pi^*c_1(U)\cdot\xi)(2\xi + \pi^*c_1(E^*) + \pi^*c_1(U)) \\ &= -\frac{3}{2}c_1(E^*) - \frac{9}{2}c_1(U) = -\frac{3}{2}\deg \Delta_g. \end{aligned}$$

Since Z and U are smooth, the general fiber of g is smooth, so Δ_g is effective, i.e., $\deg \Delta_g \leq 0$. \square

Proposition 2.2. *Let X be a smooth projective variety with $n := \dim X \geq 2$ and $f: X \rightarrow Y$ the contraction of an extremal ray on X . Suppose $\dim Y = n - 1$. Then there exists a smooth surface $S \subset X$ such that $\text{ch}_2(X).S \leq 0$.*

Proof. There exists a subvariety $A \subset Y$ of codimension ≥ 2 such that if we let $U := Y \setminus A$, then U is smooth and

$$g := f|_{f^{-1}(U)}: f^{-1}(U) \rightarrow U$$

is equidimensional. By [And85, Thm. 3.1] (cf. [Kac97, Prop. 2.2]), this implies that g is a conic bundle. Now if we take $n - 2$ general ample divisors H_1, \dots, H_{n-2} on Y , then $H_1 \cap \dots \cap H_{n-2}$ is contained in U and both $H_1 \cap \dots \cap H_{n-2}$ and

$$S := g^{-1}(H_1 \cap \dots \cap H_{n-2})$$

are smooth. Then we have $\text{ch}_2(X).S = \text{ch}_2(T_{f^{-1}(U)}|_S) \leq 0$ by Lemma 2.1. \square

3. \mathbb{P}_2 -FIBRATIONS

We now consider extremal contractions

$$f: X \rightarrow Y$$

where X is a 2-Fano fourfold and Y is a surface. We will show in this section that the above situation cannot occur.

We start by observing that a general fiber F of f is a smooth surface with $N_{F|X} \cong \mathcal{O}_F^{\oplus 2}$, so $-K_F = -K_X|_F$ is ample and $\text{ch}_2(F) = \text{ch}_2(X).S > 0$. In other words, F is 2-Fano, and thus $F \cong \mathbb{P}_2$.

By [AW97, Cor. (1.4)], Y is smooth. So, since X is Fano, Y must be a blow-up of either \mathbb{P}_2 or a Hirzebruch surface. In particular, there exists a base point free linear system on Y whose general element ℓ is isomorphic to \mathbb{P}_1 .

If we consider the preimage $X_\ell := f^{-1}(\ell)$, then X_ℓ is a smooth threefold with the property that $\text{ch}_2(X_\ell).S > 0$ for any $S \in \overline{NE}_2(X_\ell) \setminus \{0\}$.

We now apply the following lemma:

Lemma 3.1. *Let $g: Z \rightarrow T$ be a surjective morphism from a smooth projective threefold Z to a smooth curve T . Assume that*

- (1) $-K_Z$ is g -ample,
- (2) $\text{ch}_2(Z).S > 0$ for all irreducible surfaces $S \subset Z$ and
- (3) a general fiber of g is isomorphic to \mathbb{P}_2 .

Then any singular fiber F_{sing} of g can (as a divisor) be decomposed as

$$F_{\text{sing}} = F_1 + F_2 + F_3,$$

where $F_i \cong \mathbb{F}_1$ for $i = 1, 2, 3$ and $F_i \cap F_j$ is a line for $i \neq j$ (either a fiber of the ruling or the exceptional section of \mathbb{F}_1).

More precisely, there is a finite sequence of birational morphisms

$$Z = Z_0 \xrightarrow{\alpha_1} Z_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_s} Z_s,$$

where Z_s is a \mathbb{P}_2 -bundle over T and each α_i is given as the composition

$$\alpha_i = \gamma_i \circ \beta_i,$$

where β_i is the blow-up of a line in a smooth fiber of Z_i over T and γ_i is the blow-up of a non-trivial fiber of β_i .

Proof. The result is a corollary of the more general classification of Del Pezzo fibrations over curves carried out by T. Fujita in [Fuj90]. In our special case, the proof is simplified considerably by ruling out most cases using the positivity assumption on the second Chern character as we will do in the following.

We let F_{sing} be a singular fiber of g . Then by [ARM12, Thm. 5], F_{sing} is reducible. So, as a divisor, we can decompose F_{sing} as

$$F_{\text{sing}} = \sum_{j=1}^r m_j F_j,$$

where $r \geq 2$, the F_j are the irreducible components of F_{sing} and $m_j \geq 1$. Then we have the following

Lemma 3.2. *For any irreducible component F_j of F_{sing} , there exists an irreducible curve $C_j \subset F_j$ with $C_j \cdot F_j < 0$ whose numerical class $[C_j]$ is extremal in $\overline{NE}(Z/T)$.*

Proof. Since the fiber F_{sing} is connected, the irreducible component F_j meets another irreducible component F_k with $k \neq j$. Let $\tilde{C} \subset F_j$ be a curve meeting F_k but not contained in any F_i for $i \neq j$. Then clearly

$$F_k \cdot \tilde{C} > 0 \quad \text{and} \quad F_i \cdot \tilde{C} \geq 0 \quad \text{for } i \neq j.$$

Since $F_{\text{sing}} \cdot \tilde{C} = 0$, this implies

$$F_j \cdot \tilde{C} < 0.$$

Now since $-K_Z$ is g -ample, by the relative cone theorem, $\overline{NE}(Z/T)$ is generated by the classes of finitely many irreducible extremal curves, so in particular, we can write

$$\tilde{C} \equiv \sum_{l=1}^m \alpha_l \tilde{C}_l,$$

where the \tilde{C}_l are irreducible extremal curves and $\alpha_l > 0$. From $F_j \cdot \tilde{C} < 0$, it then follows that there exists an $l \in \{1, \dots, m\}$ such that $F_j \cdot \tilde{C}_l < 0$ and thus $C_j := \tilde{C}_l \subset F_j$. \square

By the relative contraction theorem, there exists for each j a relative Mori contraction of g over T contracting the $C_j \subset F_j$ constructed in the preceding lemma, i.e., a Mori contraction $\phi_j: Z \rightarrow Z'_j$, where Z'_j is a normal projective variety, and a morphism $g'_j: Z'_j \rightarrow T$ such that $g'_j \circ \phi_j = g$.

If ϕ_j contracts any curve contained in a general fiber of g , then $\phi_j = g$, thus g is itself a Mori contraction and thus a \mathbb{P}_2 -bundle by [Mor82, Thm. 3.5]. This is a contradiction to the reducibility of F_{sing} , so ϕ_j must be birational.

Thus, by [Mor82, Thm. 3.3], there is an irreducible divisor D_j on Z contained in a fiber of g such that ϕ_j contracts D_j and ϕ_j is an isomorphism on $Z \setminus D_j$. For any $i \neq j$, any curve $C' \subset F_i$ with $C' \not\subset F_j$ satisfies $C' \cdot F_j \geq 0$, so C' is not numerically proportional to C_j . In particular, ϕ_j cannot contract F_i . Thus we have shown that $D_j = F_j$, so again by [Mor82, Thm. 3.3], we obtain the following types of possible fiber components F_j :

- (T1) $F_j \cong \mathbb{P}_2$ with normal bundle $N_{F_j|Z} \cong \mathcal{O}_{\mathbb{P}_2}(-k)$, $k \in \{1, 2\}$,
- (T2) $F_j \cong \mathbb{P}_1 \times \mathbb{P}_1$ with $N_{F_j|Z} \cong \mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_1}(-1, -1)$,
- (T3) $F_j \cong Q$, where $Q \subset \mathbb{P}_3$ is a quadratic cone and $N_{F_j|Z} \cong \mathcal{O}_{\mathbb{P}_3}(-1)|_Q$,
- (T4) Z'_j is smooth and there exists a smooth curve $C'_j \subset Z'_j$ such that ϕ_j is the blow-up of C'_j . Thus, $F_j \cong \mathbb{P}(N_{C'_j|Z'_j}^*)$ and $N_{F_j|Z} \cong \mathcal{O}_{\mathbb{P}(N_{C'_j|Z'_j}^*)}(-1)$.

We further note that in types (T1), (T2) and (T3), the fiber component F_j is mapped to a point by the contraction morphism ϕ_j .

By straightforward computations, we can obtain the intersection number $\text{ch}_2(Z) \cdot F_j$ in each of these cases:

- (T1) $\text{ch}_2(Z) \cdot F_j = \frac{3}{2} + \frac{1}{2}k^2$,
- (T2) $\text{ch}_2(Z) \cdot F_j = 1$,
- (T3) $\text{ch}_2(Z) \cdot F_j = 1$,
- (T4) $\text{ch}_2(Z) \cdot F_j = -\frac{1}{2} \deg N_{C'_j|Z'_j}$.

By assumption, we have $\text{ch}_2(Z) \cdot F_j > 0$ for all j . Furthermore, since $\text{ch}_2(Z) \cdot F_{\text{sing}} = \text{ch}_2(\mathbb{P}_2) = \frac{3}{2}$, we have

$$\sum_{j=1}^r m_j \text{ch}_2(Z) \cdot F_j = \frac{3}{2}.$$

Combining this with the above calculations, we see that only the following possibilities remain (up to re-numbering the F_j):

- (P1) $r = 2$, $m_1 = 1$, F_1 is of type (T4) with $\deg N_{C'_1|Z'_1} = -1$, and $m_2 \in \{1, 2\}$ with F_2 of type (T2), (T3) or (T4).
- (P2) $r = 3$, $m_1 = m_2 = m_3 = 1$, and each F_j is of type (T4) with $\deg N_{C'_j|Z'_j} = -1$.

We now show that possibility (P1) cannot occur. We assume that a fiber F_{sing} of g satisfies (P1). We consider the contraction $\phi_1: Z \rightarrow Z'_1$ constructed above and let $F'_{\text{sing}} = \phi_1(F_{\text{sing}})$. We know that Z'_1 is smooth, which implies that g'_1 is flat. Since F'_{sing} is irreducible, we can apply [ARM12, Thm. 5] to conclude that $F'_{\text{sing}} \cong \mathbb{P}_2$. But now ϕ_1 is the blow-up of a smooth curve

$C'_1 \subset F'_{\text{sing}}$, so in particular, F_2 , which is the strict transform of F'_{sing} in Z , is isomorphic to $F'_{\text{sing}} \cong \mathbb{P}_2$, contradicting (T2), (T3) and (T4).

It remains to study possibility (P2) in greater detail. We let $F_{\text{sing}} = F_1 + F_2 + F_3$ be a fiber of g that satisfies (P2). We first note that since $-K_Z$ is g -ample, we have

$$F_i.F_j.(-K_Z) \geq 0 \quad \text{for } i \neq j$$

and since F is connected, we can for each j even find an $i_0 \neq j$ such that

$$F_{i_0}.F_j.(-K_Z) > 0.$$

Together with $F_j.(F_1 + F_2 + F_3) = 0$, this implies that

$$F_j^2.(-K_Z) < 0 \quad \text{for } j = 1, 2, 3. \quad (1)$$

By applying Kodaira vanishing and using (T4) and the adjunction formula, we obtain

$$\chi(N_{F_j|Z}) = \chi(K_{F_j} - K_Z|_{F_j}) = 0. \quad (2)$$

Riemann-Roch yields

$$\chi(N_{F_j|Z}) = \frac{1}{2}(F_j|_{F_j})^2 - \frac{1}{2}K_{F_j}.F_j|_{F_j} + \chi(\mathcal{O}_{F_j}) = \frac{1}{2}F_j^2.(-K_Z) + \chi(\mathcal{O}_{F_j}).$$

Applying (1) and (2), we obtain

$$\chi(\mathcal{O}_{F_j}) > 0.$$

Since obviously $h^2(\mathcal{O}_{F_j}) = h^0(K_{F_j}) = 0$, we conclude that $h^1(\mathcal{O}_{F_j}) = 0$ and thus the curve $C'_j \subset Z'_j$ from (T4) must be isomorphic to \mathbb{P}_1 .

We now let $\tilde{F}_j = \phi_j(F_{\text{sing}})$. Then we have the following normal bundle sequence:

$$0 \longrightarrow N_{C'_j|\tilde{F}_j} \longrightarrow N_{C'_j|Z'_j} \longrightarrow N_{\tilde{F}_j|Z'_j}|_{C'_j} \longrightarrow 0. \quad (3)$$

But now $N_{\tilde{F}_j|Z'_j}$ is trivial, so from $\deg N_{C'_j|Z'_j} = -1$, we infer that $N_{C'_j|\tilde{F}_j} \cong \mathcal{O}_{\mathbb{P}_1}(-1)$ and hence

$$N_{C'_j|Z'_j} \cong \mathcal{O}_{\mathbb{P}_1}(-1) \oplus \mathcal{O}_{\mathbb{P}_1}.$$

So in particular, $F_j \cong \mathbb{F}_1$.

We consider again the contraction $\phi_1: Z \rightarrow Z'_1$, which is the blow-up of a smooth curve $C'_1 \subset Z'_1$. If we let F'_2 and F'_3 be the images of F_2 and F_3 via ϕ_1 , respectively, we can assume that $C'_1 \subset F'_2$. Then F_2 is mapped isomorphically onto F'_2 by ϕ_1 .

We now want to apply Lemma 3.2 to Z'_1 in order to find a contraction morphism $\psi_2: Z'_1 \rightarrow Z''_2$ which contracts F'_2 . In order to do this, we have to show that $-K_{Z'_1}$ is g'_1 -ample. Since $\overline{NE}(Z/T)$ is generated by finitely many curves, the same is true for $\overline{NE}(Z'_1/T)$. So it suffices to show that $-K_{Z'_1}.C > 0$ for any curve C contained in a fiber of g'_1 . But now since ϕ_1 is the blow-up of C'_1 , we have

$$K_Z = \phi_1^*K_{Z'_1} + F_1,$$

so $-K_{Z'_1}.C > 0$ is certainly true for all irreducible $C \neq C'_1$ since such C can be obtained as the image of a curve $\tilde{C} \subset Z$ with $\tilde{C} \not\subset F_1$. But now also

$-K_{Z'_1}.C'_1 > 0$, which one can see as follows: Let $C_e \subset Z$ be the exceptional section of $F_1 \rightarrow C'_1$. Then $\phi_{1*}C_e = C'_1$ and $K_{F_1}|_{C_e} \cong \mathcal{O}_{\mathbb{P}_1}(-1)$, hence by adjunction

$$F_1.C_e = -K_Z.C_e - 1 \geq 0$$

and thus $-K_{Z'_1}.C'_1 > 0$ as in the previous case.

So by Lemma 3.2, we obtain a relative contraction morphism $\psi_2: Z'_1 \rightarrow Z''_2$ contracting F'_2 as desired. We let $g''_2: Z''_2 \rightarrow T$ be the induced morphism. Since the exceptional divisor of ψ_2 is $F'_2 \cong \mathbb{F}_1$ we conclude by [Mor82, Theorem 3.3] that Z''_2 is smooth and ψ_2 is the blow-up of a smooth curve $C''_2 \subset Z''_2$. By the same argument as in the discussion of possibility (P1) above, we see that the fiber of g''_2 over $g(F_{\text{sing}}) \in T$ must be isomorphic to \mathbb{P}_2 . If we consider the normal bundle sequence for $C''_2 \subset \psi_2(\phi_1(F_{\text{sing}})) \subset Z''_2$ (cf. sequence (3)), we can conclude from $\mathbb{F}_1 \cong F'_2 \cong \mathbb{P}(N_{C'_1|Z'_1})$ that C''_2 must be a line in \mathbb{P}_2 .

We conclude by showing that C'_1 must be a fiber of $F'_2 \rightarrow C''_2$. We first note that by adjunction, we have

$$K_{Z'_1}.C'_1 = \deg K_{C'_1} - \deg N_{C'_1|Z'_1} = -1.$$

Furthermore, obviously $F'_3 = \phi_1(F_3)$ is the strict transform of $\psi_2(F'_3) \cong \mathbb{P}_2$, so $F'_3 \cong \mathbb{P}_2$. Since $F_3 \cong \mathbb{F}_1$ is the blow-up of F'_3 in $C'_1 \cap F'_3$, we conclude that $F'_3.C'_1 = 1$ and thus $F'_2.C'_1 = -1$ since $(F'_2 + F'_3).C'_1 = 0$. Finally, since $K_{Z'_1} = \psi_2^*K_{Z''_2} + F'_2$, it follows that $\psi_2^*K_{Z''_2}.C'_1 = 0$, from which we conclude that $\psi_2(C'_1)$ must be a point since $-K_{Z''_2}$ is obviously g''_2 -ample. \square

We now apply Lemma 3.1 to X_ℓ . We get a finite sequence

$$X_\ell = Z_0 \xrightarrow{\alpha_1} Z_1 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_s} Z_s$$

of birational morphisms as described in the lemma, where $Z_s \rightarrow \ell \cong \mathbb{P}_1$ is a \mathbb{P}_2 -bundle. We now apply to Z_s the following lemma:

Lemma 3.3. *Let E be a vector bundle on \mathbb{P}_1 of rank $r \geq 2$. Then there exists a rank-2 quotient bundle F of E such that the surface*

$$S := \mathbb{P}(F) \subset \mathbb{P}(E)$$

satisfies $\text{ch}_2(\mathbb{P}(E)).S \leq 0$.

Proof. By Grothendieck, there exist integers $a_1 \leq \cdots \leq a_r$ such that

$$E \cong \mathcal{O}_{\mathbb{P}_1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}_1}(a_r).$$

We consider the quotient

$$E \twoheadrightarrow \mathcal{O}_{\mathbb{P}_1}(a_1) \oplus \mathcal{O}_{\mathbb{P}_1}(a_2) =: F$$

given by the projection onto the first two summands of E . Then we have a natural inclusion $S := \mathbb{P}(F) \subset \mathbb{P}(E)$ and if we let $\zeta := \mathcal{O}_{\mathbb{P}(E)}(1)|_{\mathbb{P}(F)}$, the normal bundle is given by

$$N_{S|\mathbb{P}(E)} \cong \zeta \otimes \pi^*E^*/F^* \cong \zeta \otimes \pi^*(\mathcal{O}_{\mathbb{P}_1}(-a_3) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}_1}(-a_r)),$$

where $\pi: S = \mathbb{P}(F) \rightarrow \mathbb{P}_1$ denotes the natural projection. We now consider the sequence

$$0 \longrightarrow T_S \longrightarrow T_{\mathbb{P}(E)}|_S \longrightarrow N_{S|\mathbb{P}(E)} \longrightarrow 0.$$

Since F has rank 2, we have $\text{ch}_2(T_S) = 0$ by [dJS06, Prop. 4.3], so we obtain from the above sequence:

$$\begin{aligned} \text{ch}_2(\mathbb{P}(E)).S &= \text{ch}_2(N_{S|\mathbb{P}(E)}) \\ &= \frac{1}{2}\zeta^2 \cdot (r-2) + \zeta \cdot \pi^* \mathcal{O}(-a_3 - \dots - a_r) \\ &= \frac{1}{2}(r-2)(a_1 + a_2) - a_3 - \dots - a_r \leq 0 \end{aligned} \quad \square$$

We let $S_s := S \subset Z_s$ be the surface constructed in Lemma 3.3. We denote by S_k , $k = 0, \dots, s-1$, the strict transform of S inside Z_k .

Claim 3.4. For any $k = 0, \dots, s$, the intersection of S_k with any smooth fiber of Z_k over ℓ is a line in \mathbb{P}_2 . Furthermore, $\text{ch}_2(Z_k).S_k \leq 0$.

To prove Claim 3.4, we argue by induction over k . The case $k = s$ follows from Lemma 3.3. Now we assume by induction that Claim 3.4 is true for some $k \geq 1$. We consider the birational morphism $\alpha_k: Z_{k-1} \rightarrow Z_k$, which by Lemma 3.1 can be decomposed as

$$\alpha_k = \gamma_k \circ \beta_k,$$

where $\beta_k: \tilde{Z}_k \rightarrow Z_k$ is the blow-up of a line L_1 contained in a smooth fiber of Z_k over ℓ and $\gamma_k: Z_{k-1} \rightarrow \tilde{Z}_k$ is the blow-up of a one-dimensional fiber $L_2 \cong \mathbb{P}_1$ of β_k . We let \tilde{S}_k be the strict transform of S_k in \tilde{Z}_k and consider different cases.

The first case is that $L_1 \not\subset S_k$. Then $L_1 \cap S_k$ is a simple point by the induction hypothesis, so in particular \tilde{S}_k is the blow-up of S_k in the point $L_1 \cap S_k$. Furthermore, by [AC12, Lem. 4.13(i)],

$$\text{ch}_2(\tilde{Z}_k).\tilde{S}_k = \text{ch}_2(Z_k).S_k - \frac{3}{2} < 0.$$

Now if we blow up L_2 in this case, we either have $L_2 \not\subset \tilde{S}_k$, whence

$$\text{ch}_2(Z_{k-1}).S_{k-1} \leq \text{ch}_2(\tilde{Z}_k).\tilde{S}_k < 0$$

by [AC12, Lem. 4.13(i)], or we have $L_2 \subset \tilde{S}_k$. But then L_2 is a (-1) -curve in \tilde{S}_k by the above considerations, so

$$\text{ch}_2(Z_{k-1}).S_{k-1} = \text{ch}_2(\tilde{Z}_k).\tilde{S}_k - \frac{3}{2} + 1 < 0$$

by [AC12, Lem. 4.13(ii)].

It remains to consider the case $L_1 \subset S_k$. Then L_1 is a fiber of S_k over ℓ , so by [AC12, Lem. 4.13(ii)],

$$\text{ch}_2(\tilde{Z}_k).\tilde{S}_k = \text{ch}_2(Z_k).S_k - 1 < 0.$$

But now since $L_1 \subset S_k$, the blow-up morphism β_k maps \tilde{S}_k isomorphically onto S_k , so in particular $L_2 \not\subset \tilde{S}_k$. This implies by [AC12, Lem. 4.13(i)] that

$$\text{ch}_2(Z_{k-1}).S_{k-1} \leq \text{ch}_2(\tilde{Z}_k).\tilde{S}_k < 0.$$

Thus we have inductively constructed an irreducible (smooth) surface $S \subset X_\ell = Z_0$ with $\text{ch}_2(X_\ell).S \leq 0$ (even $\text{ch}_2(X_\ell).S < 0$ if X_ℓ is not a \mathbb{P}_2 -bundle). Now since $N_{X_\ell|X} = (f|_{X_\ell})^*N_{\ell|Y}$, we have $\text{ch}_2(N_{X_\ell|X}) = 0$, so it follows that

$$\text{ch}_2(X).S \leq 0,$$

therefore X is not 2-Fano.

4. \mathbb{P}_3 - AND Q_3 -FIBRATIONS

In this section we deal with the case of a smooth projective fourfold X admitting an extremal contraction

$$f: X \rightarrow C$$

over a curve C . Then if X is 2-Fano, a general fiber F of f must also be 2-Fano (cf. the beginning of section 3), so either $F \cong \mathbb{P}_3$ or $F \cong Q_3$ (where $Q_3 \subset \mathbb{P}_4$ is a smooth quadric hypersurface) by [AC12, Thm. 1.2]. In the first case we easily obtain:

Proposition 4.1. *In the above situation, assume that the general fiber of f is \mathbb{P}_3 . Then X is not 2-Fano.*

Proof. Since the relative Picard number of f is 1, every fiber of f is irreducible. By [ARM12, Thm. 5], this implies that f is a \mathbb{P}^3 -bundle. Since C is a curve, there exists a rank-4 vector bundle on C such that $X \cong \mathbb{P}(E)$. The proposition now follows from [dJS06, Cor. 4.6]. \square

It remains to consider the case where $F \cong Q_3$:

Proposition 4.2. *In the above situation, assume that the general fiber of f is Q_3 . Then X is not 2-Fano.*

Proof. Since the relative Picard number of f is 1, every fiber of f is irreducible. By [Ara09, Prop. 21], this implies that f is a *geometric quadric bundle*, i.e., there exists a rank-5 bundle E on C such that X embeds into $\mathbb{P}(E)$ as a divisor of relative degree 2. Now if we assume X to be 2-Fano, $-K_X$ is ample, so for $m \gg 0$, if we take two general divisors $H_1, H_2 \in |-mK_X|$, the intersection

$$S := H_1 \cap H_2$$

is a smooth surface.

We will now calculate $\text{ch}_2(X).S$. We denote by $\pi: \mathbb{P}(E) \rightarrow C$ the natural projection and let $\xi := \mathcal{O}_{\mathbb{P}(E)}(1)$. Then

$$-K_{\mathbb{P}(E)} = 5\xi + \pi^*(\det E^* - K_C)$$

and

$$\text{ch}_2(\mathbb{P}(E)) = \frac{5}{2}\xi^2 + \pi^*c_1(E^*).\xi$$

(cf. [dJS06, Lem. 4.1]).

Since f is a geometric quadric bundle, there exists a line bundle L on C such that

$$X \in |2\xi + \pi^*L|.$$

In particular,

$$-K_X = (3\zeta + \pi^*(\det E^* - L - K_C))|_X$$

and

$$\begin{aligned} \text{ch}_2(X) &= \left(\frac{5}{2}\zeta^2 + \pi^*c_1(E^*) \cdot \zeta - \frac{1}{2}(2\zeta + \pi^*L)^2\right)|_X \\ &= \left(\frac{1}{2}\zeta^2 + \pi^*(c_1(E^*) - 2L) \cdot \zeta\right)|_X. \end{aligned}$$

We thus obtain

$$\begin{aligned} \text{ch}_2(X) \cdot S &= \text{ch}_2(X) \cdot (-mK_X)^2 \\ &= m^2 \left(\left(\frac{1}{2}\zeta^2 + \pi^*(c_1(E^*) - 2L) \cdot \zeta \right) \cdot (3\zeta + \pi^*(c_1(E^*) - L - K_C))^2 \right)|_X \\ &= m^2 \left(\frac{9}{2}\zeta^4 + \pi^*(12c_1(E^*) - 21L - 3K_C) \cdot \zeta^3 \right) \cdot (2\zeta + \pi^*L) \\ &= m^2 \left(9\zeta^5 + \pi^*(24c_1(E^*) - \frac{75}{2}L - 6K_C) \cdot \zeta^4 \right) \\ &= m^2 (15 \deg E^* - \frac{75}{2} \deg L - 6 \deg K_C). \end{aligned}$$

Since X is given by a section

$$s \in H^0(S^2E \otimes L) \subset \text{Hom}(E^*, E \otimes L),$$

the discriminant locus of f is a divisor

$$\Delta_f \in |2 \det E + 5L|.$$

We can thus rewrite the result of the above calculation as

$$\text{ch}_2(X) \cdot S = -\frac{15}{2}m^2(\deg \Delta_f + \frac{4}{5} \deg K_C).$$

We now argue by contradiction: We assume that $\text{ch}_2(X) \cdot S > 0$. Then since Δ_f is effective, we must have $\deg K_C < 0$, i.e., $C \cong \mathbb{P}_1$, and so

$$\deg \Delta_f \in \{0, 1\}.$$

By tensorizing E with a suitable line bundle, we can assume

$$0 \leq \deg E \leq 4.$$

In the case $\deg \Delta_f = 0$, we thus obtain $\deg E = \deg L = 0$. We first show that the line bundle

$$\zeta \otimes \pi^*\mathcal{O}_{\mathbb{P}_1}(-1)$$

cannot have a section: Suppose the contrary, then we get an effective Divisor

$$D \in |\zeta - \pi^*\mathcal{O}_{\mathbb{P}_1}(1)|$$

on X . Intersecting D with a general member of $|-mK_X|$ gives a surface in X and we have

$$\begin{aligned} \text{ch}_2(X) \cdot D \cdot (-mK_X) &= m\zeta^3 \cdot (\zeta - \pi^*\mathcal{O}_{\mathbb{P}_1}(1)) \cdot (3\zeta + \pi^*\mathcal{O}_{\mathbb{P}_1}(2)) \\ &= m(3\zeta^5 - \pi^*\mathcal{O}_{\mathbb{P}_1}(1) \cdot \zeta^4) = -m, \end{aligned}$$

which contradicts X being 2-Fano. So $H^0((\zeta \otimes \pi^*\mathcal{O}_{\mathbb{P}_1}(-1))|_X) = 0$ and from the exact sequence

$$0 \longrightarrow \zeta^* \otimes \pi^*\mathcal{O}_{\mathbb{P}_1}(-1) \longrightarrow \zeta \otimes \pi^*\mathcal{O}_{\mathbb{P}_1}(-1) \longrightarrow (\zeta \otimes \pi^*\mathcal{O}_{\mathbb{P}_1}(-1))|_X \longrightarrow 0$$

we conclude that also $H^0(\zeta \otimes \pi^*\mathcal{O}_{\mathbb{P}_1}(-1)) = 0$. But this means that

$$H^0(E \otimes \mathcal{O}_{\mathbb{P}_1}(-1)) = 0.$$

Since E splits as a direct sum of line bundles, we conclude that $E \cong \mathcal{O}_{\mathbb{P}_1}^{\oplus 5}$ and thus $X \cong \mathbb{P}_1 \times Q_3$ is not 2-Fano.

In the case $\deg \Delta_f = 1$, we have $\deg E = 3$ and $\deg L = -1$, so we obtain

$$-K_X = 3\zeta|_X,$$

thus X has index 3, so X is not 2-Fano by [AC12, Thm. 1.3]. \square

5. THE BIRATIONAL CASE

In this section we study the case of a birational extremal contraction

$$f: X \rightarrow Y,$$

where X is a 2-Fano fourfold. Unfortunately, we do not have a general classification result for this situation. We start by studying the blow-up of a smooth surface inside a smooth fourfold:

Lemma 5.1. *Let $p: X \rightarrow Z$ be the blow-up of a smooth fourfold Z along a smooth surface $W \subset Z$. Then, for any smooth curve $C \subset W$, the smooth surface $S := p^{-1}(C) \subset X$ satisfies*

$$\mathrm{ch}_2(X).S = -\frac{1}{2} \deg N_{W|Z}|_C.$$

Proof. Since S is a \mathbb{P}_1 -bundle, we have $\mathrm{ch}_2(S) = 0$ by [dJS06, Prop. 4.3]. The sequence

$$0 \longrightarrow T_S \longrightarrow T_X|_S \longrightarrow N_{S|X} \longrightarrow 0.$$

then implies that

$$\mathrm{ch}_2(X).S = \mathrm{ch}_2(N_{S|X}).$$

Now if we denote by E the exceptional divisor of p , we have an exact sequence

$$0 \longrightarrow N_{S|E} \longrightarrow N_{S|X} \longrightarrow N_{E|X}|_S \longrightarrow 0.$$

Since $N_{S|E} = (p|_S)^* N_{C|W}$, we have $\mathrm{ch}_2(N_{S|E}) = 0$, so it remains to calculate $\mathrm{ch}_2(N_{E|X}|_S)$. To this end, we observe that $N_{E|X} = \mathcal{O}_E(-1)$, where $\mathcal{O}_E(1)$ is the tautological line bundle on $E \cong \mathbb{P}(N_{W|Z}^*)$. Thus we obtain

$$\mathrm{ch}_2(N_{E|X}|_S) = \frac{1}{2}(\mathcal{O}_E(-1))^2 \cdot ((p|_E)^* \mathcal{O}_W(C)) = \frac{1}{2} c_1(N_{W|Z}^*) \cdot c_1(\mathcal{O}_W(C)). \quad \square$$

We can use this Lemma to obtain the following result:

Proposition 5.2. *Let $f: X \rightarrow Y$ be a birational divisorial extremal contraction which maps its exceptional divisor $D \subset X$ to a point in Y . Then X is not 2-Fano.*

Proof. We assume that X is 2-Fano.

As in [Tsu06, Prop. 5], we can choose an extremal ray $\mathbb{R}_+[C_0]$ on X such that $D.C_0 > 0$. Then by the proof of [Tsu06, Prop. 5] (cf. also [Cas09, Prop. 3.1]), the associated extremal contraction $g: X \rightarrow Z$ is either a conic bundle or the blow-up of a smooth surface inside a smooth Fano fourfold. We already showed in Proposition 2.2 that the conic bundle case cannot occur, so we conclude that Z must be a Fano fourfold and g is the blow-up of some smooth surface $W \subset Z$.

We denote by $E \subset X$ the exceptional divisor of g . Then we have $E \cong \mathbb{P}(N_{W|Z}^*)$ with the tautological line bundle $\mathcal{O}_E(1) := \mathcal{O}_{\mathbb{P}(N_{W|Z}^*)}(1)$ and the natural projection map

$$\pi: E \rightarrow W.$$

In the case $\rho(X) \geq 3$ this situation has been studied in [Fuj12]. Fujita gives a very explicit description of the possible configurations that can occur, but for our purposes it is sufficient to notice that from his Thm. 1.1 it easily follows that $\det N_{W|Z}$ is always ample, which contradicts our Lemma 5.1.

So for the rest of the proof we can assume that $\rho(X) = 2$. We consider the scheme-theoretic intersection

$$V := D \cap E.$$

Claim 5.3. V is a reduced section of π , i.e., $\pi|_V$ is an isomorphism.

If we assume Claim 5.3 to be true, we immediately obtain that D and E intersect transversally and that $g|_D: D \rightarrow g(D) =: D'$ is an isomorphism. Now since f is the contraction of an extremal ray, we have

$$\mathrm{Im}(\mathrm{Pic} X \rightarrow \mathrm{Pic} D) \cong \mathbb{Z} \quad (4)$$

(cf. [And85, Thm. 2.1]). This implies in particular that $V = E|_D$ is an ample Cartier divisor on D and thus that W is an ample Cartier divisor on $D' = g(D)$.

Since $\rho(X) = 2$, it follows that $\rho(Z) = 1$, so that D' is an ample divisor on Z . The adjunction formula now gives

$$\mathcal{O}_W(K_W) = (\omega_{D'} \otimes \mathcal{O}_{D'}(W))|_W = \mathcal{O}_{D'}(K_Z + D' + W)|_W. \quad (5)$$

But since Z and W are smooth, we also have

$$\mathcal{O}_W(K_W) = \mathcal{O}_W(K_Z) \otimes \det N_{W|Z}. \quad (6)$$

Comparing (5) and (6), we obtain

$$\det N_{W|Z} = \mathcal{O}_{D'}(D' + W)|_W,$$

thus $\det N_{W|Z}$ is ample by our previous considerations. This contradicts Lemma 5.1.

It remains to prove Claim 5.3. To do this, we need to obtain more information about W . We first note that since X is 2-Fano, we have

$$\det N_{W|Z}^* \cdot C \geq 1 \quad \text{for any curve } C \subset W$$

by Lemma 5.1. Since Z is Fano, this implies by (6) that $-K_W$ is ample. Furthermore, (6) gives

$$-K_W \cdot C \geq 2 \quad \text{for any curve } C \subset W.$$

In particular, W cannot contain any (-1) -curve, so we conclude that either $W \cong \mathbb{P}_2$ or $W \cong \mathbb{P}_1 \times \mathbb{P}_1$.

The map f induces a contraction on E contracting V to a point. We now consider the reduction

$$\tilde{V} := V_{\mathrm{red}} = (D \cap E)_{\mathrm{red}}.$$

The restriction $g|_{\tilde{V}}$ is then a finite morphism, so \tilde{V} is a multisection of π . Now let $\ell \subset W$ be a general line. Then $f|_{\pi^{-1}(\ell)}$ must be the contraction of the minimal section of the Hirzebruch surface $\pi^{-1}(\ell)$. This implies that \tilde{V} is indeed a section of π . Furthermore, since $N_{\tilde{V}|E}^*$ is ample, it follows that $N_{\tilde{V}|E}^*$ is ample.

Thus it only remains to show that V is reduced. As a divisor on E , we can write

$$V = \lambda \tilde{V} \quad \text{for some } \lambda \geq 1.$$

Claim 5.3 is then proved if we show that $\lambda = 1$. In order to show this, we first use Ando's result on the classification of the general fiber of the exceptional divisor of an extremal contraction [And85, Thm. 2.1]. By this result we have one of the following cases:

- (1) $D \cong \mathbb{P}_3$,
- (2) $D \cong Q_3$, or
- (3) D is a Del Pezzo variety such that if A is an ample generator of $\text{Im}(\text{Pic } X \rightarrow \text{Pic } D) \cong \mathbb{Z}$, then

$$\omega_D \cong (A^*)^{\otimes 2}. \quad (7)$$

In the first case ($D \cong \mathbb{P}_3$), Tsukioka proves in [Tsu06, Lem. 2] that $\lambda = 1$, but his argument also applies to the second case ($D \cong Q_3$): If $\lambda > 1$, then $V = D \cap E$ is singular in every point of \tilde{V} , so in particular $T_D|_{\tilde{V}} \cong T_E|_{\tilde{V}}$ (observe that D is smooth). This implies that $N_{\tilde{V}|D} \cong N_{\tilde{V}|E}$. This is a contradiction since $N_{\tilde{V}|D}$ is ample because $\rho(D) = 1$, but $N_{\tilde{V}|E}$ is negative as seen above.

It remains to study the Del Pezzo case. Since by adjunction,

$$\omega_D \cong \mathcal{O}_D(K_X + D),$$

we have

$$\mathcal{O}_D(K_X) \cong \mathcal{O}_D(D) \quad (8)$$

by (7). We thus get

$$\mathcal{O}_{\tilde{V}}(D) \cong \mathcal{O}_{\tilde{V}}(K_X) \cong \mathcal{O}_{\tilde{V}}(g^*K_Z + E) \cong \mathcal{O}_E(-1)|_{\tilde{V}} \otimes \mathcal{O}_W(K_Z). \quad (9)$$

On the other hand, we have

$$\mathcal{O}_{\tilde{V}}(D) \cong \mathcal{O}_E(V)|_{\tilde{V}} \cong (\mathcal{O}_E(\tilde{V})|_{\tilde{V}})^{\otimes \lambda},$$

i.e., $\mathcal{O}_{\tilde{V}}(D)$ is divisible by λ .

Now since \tilde{V} is a section of π , it is given by an element in $H^0(\mathcal{O}_E(1) \otimes \pi^*L)$ for some $L \in \text{Pic } W$, or, equivalently, by a section

$$s \in H^0(N_{W|Z}^* \otimes L)$$

without zeroes. The section s induces a short exact sequence of vector bundles

$$0 \longrightarrow L^* \longrightarrow N_{W|Z}^* \longrightarrow \det N_{W|Z}^* \otimes L \longrightarrow 0, \quad (10)$$

from which we conclude that

$$\mathcal{O}_E(1)|_{\tilde{V}} \otimes L \cong N_{\tilde{V}|E} \cong \det N_{W|Z}^* \otimes L^{\otimes 2}. \quad (11)$$

This implies together with (9) that

$$\mathcal{O}_{\tilde{V}}(D) \cong \mathcal{O}_W(K_Z) \otimes \det N_{W|Z} \otimes L^* \cong \mathcal{O}_W(K_W) \otimes L^*. \quad (12)$$

Since $\mathcal{O}_D(-D)$ is ample, we can conclude that $\mathcal{O}_W(-K_W) \otimes L$ is an ample line bundle.

We now first consider the case $W \cong \mathbb{P}_2$. Then $-K_W = \mathcal{O}_{\mathbb{P}_2}(3)$. Since Z is Fano and Lemma 5.1 holds, only two cases can occur by (6):

- (1) $\mathcal{O}_W(-K_Z) \cong \mathcal{O}_{\mathbb{P}_2}(1)$ and $\det N_{W|Z}^* \cong \mathcal{O}_{\mathbb{P}_2}(2)$, or
- (2) $\mathcal{O}_W(-K_Z) \cong \mathcal{O}_{\mathbb{P}_2}(2)$ and $\det N_{W|Z}^* \cong \mathcal{O}_{\mathbb{P}_2}(1)$.

If we let

$$L \cong \mathcal{O}_{\mathbb{P}_2}(a),$$

then we have $a \geq -2$ because $L - K_W$ is ample as shown above. We already showed above that $N_{\tilde{V}|E}^* \cong \det N_{W|Z} \otimes (L^*)^{\otimes 2}$ is ample, which yields $\deg N_{W|Z} - 2a > 0$, i.e., $a < \frac{1}{2} \deg N_{W|Z}$. In the two cases stated above, this means the following:

- (1) In this case, we have $-2 \leq a < -1$, so $a = -2$ and thus by (12)

$$\mathcal{O}_{\tilde{V}}(D) \cong \mathcal{O}_{\mathbb{P}_2}(1).$$

Since $\mathcal{O}_{\tilde{V}}(D)$ is divisible by λ , we get $\lambda = 1$.

- (2) Here we must have $-2 \leq a < -\frac{1}{2}$, so $a \in \{-2, -1\}$. If $a = -2$, we get $\lambda = 1$ as in the first case. If $a = -1$, we get

$$\mathcal{O}_E(-1)|_{\tilde{V}} \cong \mathcal{O}_{\tilde{V}}$$

by (11). This is a contradiction because

$$\mathcal{O}_E(-1)|_{\tilde{V}} \cong \mathcal{O}_E(E)|_{\tilde{V}} \cong \mathcal{O}_D(E)|_{\tilde{V}}$$

is ample by (4)

It remains to consider the case $W \cong \mathbb{P}_1 \times \mathbb{P}_1$. Then $-K_W \cong \mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_1}(2, 2)$ and thus by (6), we must have

$$\mathcal{O}_W(-K_Z) \cong \det N_{W|Z}^* \cong \mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_1}(1, 1)$$

(again using Lemma 5.1 and the fact that Z is Fano). If we let

$$L \cong \mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_1}(a, b)$$

and use again the fact that $L - K_W$ is ample, we have $a, b \geq -1$. From the ampleness of $N_{\tilde{V}|E}^*$ we obtain $-1 - 2a > 0$ and $-1 - 2b > 0$, so $a = b = -1$ and thus by (12)

$$\mathcal{O}_{\tilde{V}}(D) \cong \mathcal{O}_{\mathbb{P}_1 \times \mathbb{P}_1}(1, 1),$$

from which we again conclude $\lambda = 1$. □

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